

Exam sets October 2025 Solutions

Always explain your answers. It is allowed to refer to definitions, lemmas and theorems from the lecture notes but not to other sources. All questions are independent and count equally so make sure you try each of them. Good luck!

0. Prove that $[4] \setminus ([3] \setminus [2]) \neq ([4] \setminus [3]) \setminus [2]$.
Define $L = [4] \setminus ([3] \setminus [2])$ and $R = ([4] \setminus [3]) \setminus [2]$. We claim $0 \in L$ and $0 \notin R$ from which it follows that $L \neq R$. To see that $0 \in L$ we note that $0 \in [4]$ while $0 \notin [3] \setminus [2]$ because $0 \in [3] \cap [2]$. Also $0 \notin R$ because $0 \in [4] \cap [3]$ so $0 \notin [4] \setminus [3] \supset R$.
1. Given an injective function $f : X \rightarrow Y$ and a subset $A \subseteq X$, prove that A and $f(A)$ have the same cardinality.
Define $F : f(A) \rightarrow A$ by $F(y) = a$ where a is the unique (by injectivity) element of A such that $f(a) = y$. Then F is invertible with inverse $g : A \rightarrow f(A)$ defined by $g(a) = f(a)$. To check that $g = F^{-1}$ we compute for $a \in A$ that $F(g(a)) = F(f(a)) = a$ by definition of F . Conversely for any $y \in f(A)$ we have $g(F(y)) = g(a)$, where $f(a) = y$ and by definition of g we conclude $g(F(y)) = g(a) = f(a) = y$.
2. Give a concrete example of a set X and a surjective function $f : X \rightarrow X$ that is NOT injective and prove your claims.
The set needs to be infinite for this to work. A simple example would be $X = \mathbb{N}$ and f defined by $f(x) = \frac{x+(-1)^x x}{4}$. This is surjective because for all $x \in \mathbb{N}$ we have $x = f(2x)$. It is not injective since $f(2x+1) = 0$ for all $x \in \mathbb{N}$.
3. Define a relation $\sim$ on $\mathbb{N}$ as follows: $x \sim y$ iff $-10 \leq x - y \leq 10$. Is $\sim$ an equivalence relation? If yes prove it, if no explain.
No because it is not transitive. For example take $x = 0, y = 10, z = 20$. Then $x \sim y$ and $y \sim z$ because $x - y = -10 = y - z$. On the other hand $x \sim z$ is not true because $x - z = -20 < -10$.
4. For a fixed $n \in \mathbb{N}$ such that $n \geq 2$, we say a set X is n -good iff the following happens. Whenever we have subsets $X_0, X_1, \dots, X_{n-1} \subseteq X$ such that for all $i \neq j \in [n]$ we have $X_i \neq X_j$, then $\bigcap_{j=0}^{n-1} X_j = \emptyset$. Is it true that $[n]$ is n -good for all $n > 2$? Prove or give a counter example.
No. For $n = 3$ we can take $X_0 = 0, X_1 = 0, 1$ and $X_2 = 0, 2$. Then all three sets are distinct but their triple intersection is non-empty, it is $X_0 \cap X_1 \cap X_2 = \{0\} \neq \emptyset$.
5. Prove by induction that for all $n \in \mathbb{N}$:

$$\# \left(\bigcup_{j=0}^n ([j] \times \{j\}) \right) = \frac{n(n+1)}{2}$$

Define for any $n \in \mathbb{N}$ the statement S_n to be $\# \left(\bigcup_{j=0}^n ([j] \times \{j\}) \right) = \frac{n(n+1)}{2}$. Notice that S_0 is true because $[0] \times \{0\} = \emptyset$ which has cardinality $0 = 0(0+1)/2$. Now assume S_n holds for some n . To prove S_{n+1} we write $\bigcup_{j=0}^{n+1} ([j] \times \{j\}) = A \cup B$, where $A = [n+1] \times \{n+1\}$ and $B = \bigcup_{j=0}^n ([j] \times \{j\})$. $A \cap B = \emptyset$ because these are sets of ordered pairs of integers and the second coordinate of all members of B is less than $n+1$, while it is precisely $n+1$ for all elements of A . By a lemma in Chapter 2 of the lecture notes $\#(A \cup B) = \#A + \#B$ is true. By the induction hypothesis we conclude that $\#A + \#B = n+1 + n(n+1)/2 = (2+n)(n+1)/2$ proving S_{n+1} . This concludes our induction proof.

6. Consider the sets $E_0 = \{\odot, \ominus\}$, $E_1 = \{\mathbb{N}^{\mathbb{N}}\}$ and $E_2 = \{\text{cat}, \text{dog}\}$ and the equivalence relation $\sim$ on $X = E_0 \cup E_1 \cup E_2$ defined by $x \sim y$ iff there is an $i \in [3]$ such that $\{x, y\} \subseteq E_i$. Write down an explicit system of representatives $R \subseteq X$ for the relation $\sim$ and explain your answer.
We claim that $X/\sim = \{E_0, E_1, E_2\}$ and thus a system of representatives would amount to choosing one element from each of the E_i , for example $R = \{\odot, \mathbb{N}^{\mathbb{N}}, \text{cat}\}$. To verify our claim on $X/\sim$ notice that if $x, y \in E_i$ for some i then $\{x, y\} \subseteq E_i$ and conversely if $x \in E_i$ and $y \in E_j$ with $i \neq j$ then there is no $k \in [3]$ such that $\{x, y\} \subseteq E_k$ because $E_r \cap E_s = \emptyset$ for all $r, s \in [3]$.
7. Explain what is wrong with the following reasoning: Define an equivalence relation $\sim$ on $\mathbb{Q}$ by saying $p \sim q$ iff $\{r \in \mathbb{Q} : p \leq r \leq q\} \cup \{r \in \mathbb{Q} : q \leq r \leq p\} \neq \emptyset$. Since $\mathbb{Q}$ is infinite we can choose two distinct elements $\bar{x} \neq \bar{y} \in \mathbb{Q}/\sim$. If $x \leq y$ then $x \leq \frac{x+y}{2} \leq y$ so that $x \sim y$ and if $y \leq x$ the same conclusion holds. This is a contradiction because $x \sim y$ means $\bar{x} = \bar{y}$ contrary to our assumption.
The mistake is the assumption that the number of equivalence classes for an equivalence relation on an infinite set must contain at least two distinct elements. In this case all fractions become equivalent.
8. For an uncountable set G define $X = G \cup \{0\}$ and recall $C_0 \in X^X$ is the constant function satisfying $C_0(x) = 0$ for all $x \in X$. Write down the range of the function $C_0 \circ \text{id}_X$ and also the range of the function $\text{id}_X \circ C_0$. Since composition with the identity leaves any function unchanged all we need to do is determine the range of C_0 . Since C_0 is constant we have $C_0(X) = \{0\}$.
9. For any set G prove that $(G^G)^G$ has the same cardinality as $G^{G \times G}$.
Define $F : (G^G)^G \rightarrow G^{G \times G}$ by $F(f)(x, y) = f(x)(y)$ and $H : G^{G \times G} \rightarrow (G^G)^G$ by $H(j)(x)(y) = j(x, y)$. Then $F(H(j))(x, y) = H(j)(x)(y) = j(x, y)$ and $H(F(f))(x)(y) = F(f)(x, y) = f(x)(y)$ as required to show that $H = F^{-1}$. It follows that domain and codomain of F have the same cardinality.